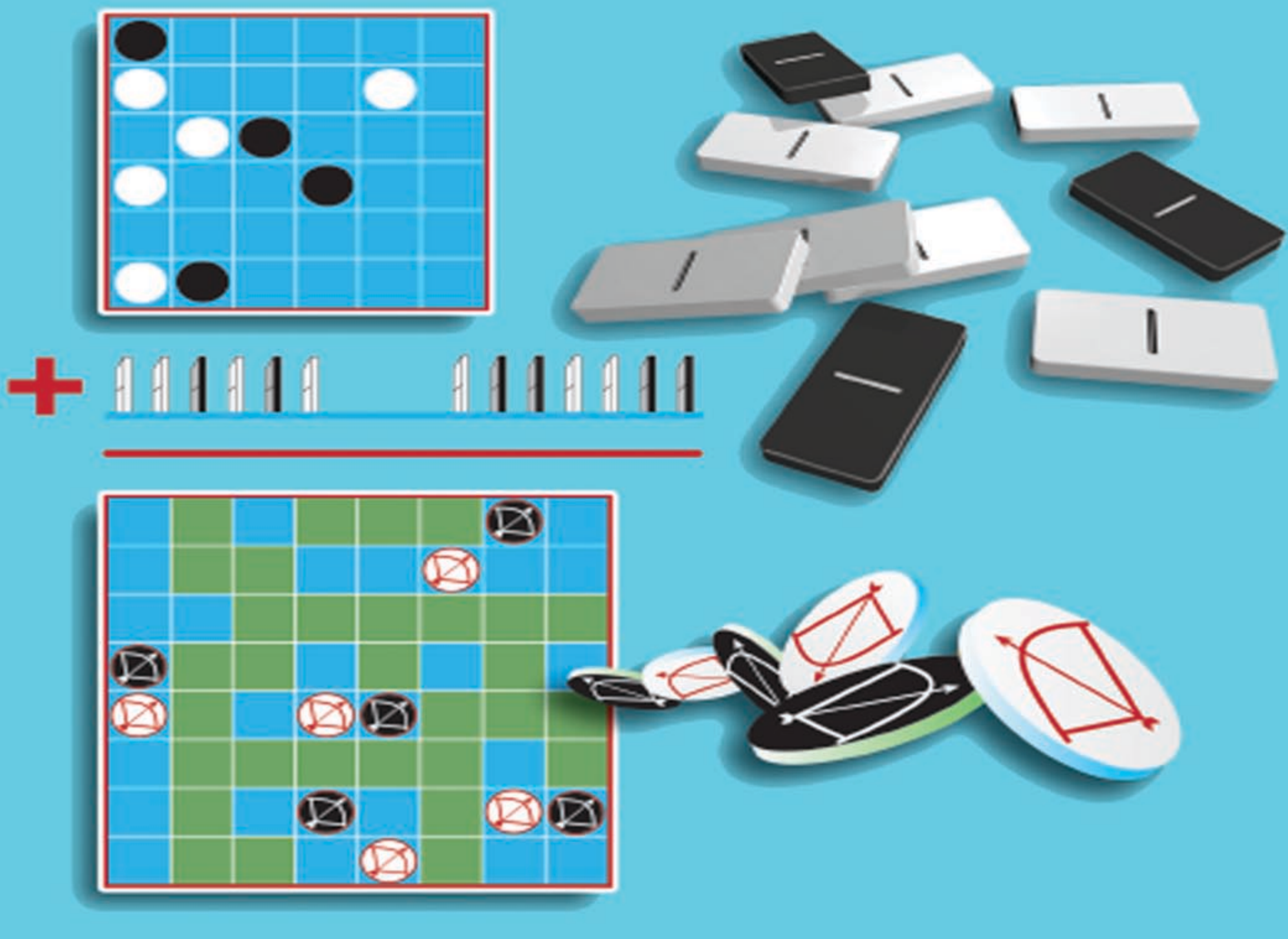
Second Edition

LESSONS IN PLAY

An Introduction to Combinatorial Game Theory



Michael H. Albert • Richard J. Nowakowski • David Wolfe

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AN A K PETERS BOOK

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To Richard K. Guy, a gentleman and a mathematician

Contents

Preface

It should be noted that children's games are not merely games. One should regard them as their most serious activities.

Michel Eyquem de Montaigne

Herein we study games of pure strategy, in which there are only two players[1] who alternate moves, without using dice, cards, or other random devices, and where the players have perfect information about the current state of the game. Familiar games of this type include TIC TAC TOE, DOTS & BOXES, CHECKERS, and CHESS. Obviously, card games such as GIN RUMMY and dice games such as BACKGAMMON are not of this type. The game of BATTLESHIP has alternate play and no chance elements, but fails to include perfect information — in fact, that's rather the point of BATTLESHIP. The games we study have been dubbed *combinatorial games* to distinguish them from the games usually found under the heading of *game theory,* which are games that arise in economics and biology.

For most of history, the mathematical study of these games consisted largely of separate analyses of extremely simple games. This was true up until the 1930s when the Sprague-Grundy theory provided the beginnings of a mathematical foundation for a more general study of games. In the 1970s, the twin tomes *On Numbers and Games* by Conway and *Winning Ways* by Berlekamp, Conway, and Guy established and publicized a complete and deep theory, which can be deployed to analyze countless games. One cornerstone of the theory is the notion of a disjunctive sum of games, introduced by John Conway for normal-play games. This scheme is particularly useful for games that split naturally into components. *On Numbers and Games* describes these mathematical ideas at a sophisticated level. *Winning Ways* develops these

[1] In 1972, Conway's first words to one of the authors, who was an undergraduate at the time, was "What's $1 + 1 + 1$?" alluding to three-player games. This question has still not been satisfactorily answered.

ideas, and many more, through playing games with the aid of many a pun and witticism. Both books have a tremendous number of ideas, and we acknowledge our debt to the books and to the authors for their kind words and teachings throughout our careers.

The goal of our book is less grand in scale than either of the two tomes. We aim to provide a guide to the evaluation scheme for normal-play, two-player, finite games. The guide has two threads, the theory and the applications.

The theory is accessible to any student who has a smattering of general algebra and discrete mathematics. Generally, this means a third-year college student, but any good high school student should be able to follow the development with a little help. We have attempted to be as complete as possible, though some proofs in the latter chapters have been omitted, because the theory is more complex or is still in the process of being developed. Indeed, in the last few months of writing the first edition, Conway prevailed on us to change some notation for a class of all-small games. This *uptimal* notation turned out to be very useful, and it makes its debut in this book.

We have liberally laced the theory with examples of actual games, exercises and problems. One way to understand a game is to have someone explain it to you; a better way is to think about it while pushing some pieces around; and the best way is to play it against an opponent. Completely solving a game is generally hard, so we often present solutions to only some of the positions that occur within a game. The authors invented more games than they solved during the writing of this book. While many found their way into the book, most of these games never made it to the rulesets found at the end. A challenge for you, the reader of our missive, and as a test of your understanding, is to create and solve your own games as you progress through the chapters.

Since the first appearance of *On Numbers and Games* and *Winning Ways,* there have been several conferences specifically on combinatorial games. The subject has moved forward and we present some of these developments. However, the interested reader will need to read further afield to find the theories of loopy games, misère-play games, other (non-disjunctive) sums of games, and the computer science approach to games. The proceedings of these conferences [Guy91, Now96, Now02, AN07, Now15, FN04] would be good places to start.

Organization of the Book

The main idea of the part of the theory of combinatorial games covered in this book is that it is possible to assign values to games. These values, which are not simply numbers, can be used to replace the actual games when deciding who wins and what the winning strategies might be.

Each chapter has a prelude that includes problems for the student to use as a warm-up for the mathematics to be found in the following chapter. The prelude also contains guidance to the instructor for how one can wisely deviate from the material covered in the chapter.

Exercises are sprinkled throughout each chapter. These are intended to reinforce, and check the understanding of, the preceding material. Ideally then, a student should try every exercise as it is encountered. However, there should be no shame associated with consulting the solutions to the exercises found at the back of the book if one or more of them should prove to be intractable. If that still fails to clear matters up satisfactorily, then it may be time to consult a *games guru*.

Chapter 0 introduces basic definitions and loosely defines that portion of game theory which we address in the book. Chapter 1 covers some general strategies for playing or analyzing games and is recommended for those who have not played many games. Others can safely skim the chapter and review sections on an as-needed basis while reading the body of the work. Chapters 2, 4, and 5 contain the core of the general mathematical theory. Chapter 2 introduces the first main goal of the theory, that being to determine a game's *outcome class* or who should win from any position. Curiously, a great deal of the structure of some games can be understood solely by looking at outcome classes. Chapter 3 motivates the direction that the theory takes next. Chapters 4, 5, and 6 then develop this theory (i.e., assigning values and the consequences of these values).

Chapters 7, 8, and 9 look at specific parts of the universe of combinatorial games, and as a result, these are a little more challenging but also more concrete since they are tied more closely to actual games. Chapter 7 takes an in-depth look at *impartial* games. The study of these games pre-dates the full theory. We place them in the new context and show some of the new classes of games under present study.

Chapters 8 through 10 provide techniques for identifying and exploiting the most significant information about a game when a complete analysis might be complex and therefore unhelpful. Indeed, these are areas that have seen the most advances of late. Chapter 8 addresses hot games, games such as GO and AMAZONS in which there is a great incentive to move first, while Chapter 9 addresses *all-small* games, where the value of a move is more subtle. Chapter 10, entitled "Trimming Game Trees," describes two more recent techniques for identifying the core features of games, *reduced canonical form* and *ordinal sums*.

Chapter ω is a brief listing of other areas of active research that we could not fit into an introductory text.

In Appendix A, we present top-down induction, an approach that we use often in the text. While the student need not read the appendix in its entirety, the first few sections will help ground the format and foundation of the inductive proofs found in the text.

Appendix B is a brief introduction to CGSuite, a powerful programming toolkit written by Aaron Siegel in Java for performing algebraic manipulations on games. CGSuite is to the combinatorial game theorist what Maple or Mathematica is to a mathematician or physicist. While the reader need not use CGSuite while working through the text, the program does help to build intuition, double-check work done by hand, develop hypotheses, and handle some of the drudgery of rote calculations.

Appendix D contains the rules to many games, ordered alphabetically. In particular we include any game that appears multiple times in the text, or is found in the literature. We do not always state the detailed rules of a game in the text, so the reader will want to refer to this appendix often.

The supporting website for the book is located at www.lessonsinplay.com. Look there for links, programs, and addenda, as well as instructions for accessing the online solutions manual for instructors.

Acknowledgments

While we are listed as the *authors* of this text, we do not claim to be the main contributors. The textbook emerged from a mathematically rich environment created by others. We got to choose the words and consequently, despite the best efforts of friends and colleagues, all the errors are ours.

Many of the contributors to this environment are cited within the book. There were many who also contributed to and improved the contents of the text itself and who deserve special thanks. We are especially grateful to Elwyn Berlekamp, John Conway, and Richard Guy who encouraged — and, at times, hounded — us to complete the text, and we hope it helps spawn a new generation of active aficionados.

Naturally, much of the core material and development is a reframing of material in *Winning Ways* and *On Number and Games.* We have adopted some of the proofs of J P Grossman, particularly that of the *Number-Avoidance Theorem.* Aviezri Fraenkel contributed the *Fundamental Theorem of Combinatorial Games,* which makes its appearance at the start of Chapter 2. Dean Hickerson helped us to prove Theorem 6.15 on page 147, that a game with negative incentives must be a number. John Conway repeatedly encouraged us to adopt the *uptimal* notation in Chapter 9, and it took us some time to see the wisdom of his suggestions. Elwyn Berlekamp and David Molnar contributed some fine problems. Paul Ottaway, Angela Siegel, Meghan Allen, Fraser Stewart, and Neil McKay were students who pretested portions of the book and provided useful feedback, corrections, and clarifications. Elwyn Berlekamp, Richard Guy, Aviezri Fraenkel, and Aaron Siegel edited various chapters of our work for technical content, while Christine Aikenhead edited for language. Brett Stevens and Chris Lewis read and commented on parts of the book. Susan Hirshberg contributed the title of our book.

We also thank all those who identified typos and errors in our first edition, especially Matthew Ferland, Ted Hwa, Ishihara Toru, and Mike Fisher.

In this age of large international publishers, A K Peters was a fantastic and refreshing publishing house to work with on the first edition. They cared more about the dissemination of fine works than about the bottom line. We will miss them. But, we have been delighted to work with CRC Press and Taylor & Francis on this edition, who have been tremendously helpful, especially given the authors' predilection to procrastination.

The authors would like to thank their spice[2] for their loving support, and Lila and Tovia, who are the real *Lessons in Play*.

[2]As in *spice of our life* — more affectionate than partner, spouse, or significant other.

Preparation for Chapter 0

Before each chapter are several quick prep problems that are worth tackling in preparation for reading the chapter.

Prep Problem 0.1. Make a list of all the two-player games you know of and classify each one according to whether or not it uses elements of chance (e.g., dice, coin flips, randomly dealt cards) and whether or not there is hidden information.

Prep Problem 0.2. Locate the textbook website, www.lessonsinplay.com, and determine whether it might be of use to you.

To the instructor: Before each chapter, we will include a few suggestions to the instructor. Usually these will be examples that do not appear in the book, but that may be worth covering in lecture. The student unsatisfied by the text may be equally interested in seeking out these examples.

We highly recommend that the instructor and the student read Appendix A on top-down induction. We present induction in a way that will be unfamiliar to most, but that leads to more natural proofs, particularly those found in combinatorial game theory.

The textbook website, www.lessonsinplay.com, has directions for how instructors can obtain a solution manual.

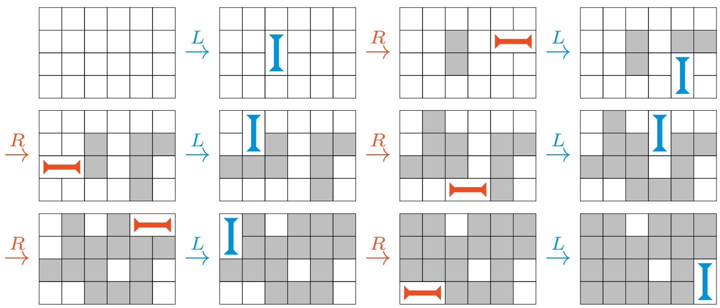
Chapter 0

Combinatorial Games

We don't stop playing because we grow old;
we grow old because we stop playing.

George Bernard Shaw

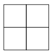
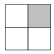
This book is all about *combinatorial games* and the mathematical techniques that can be used to analyze them. One of the reasons for thinking about games is so that you can play them more skilfully and with greater enjoyment; so let's begin with an example called DOMINEERING. To play you will need a chessboard and a set of dominoes. The domino pieces should be big enough to cover or partially cover two squares of the chessboard but no more. You can make do with a chessboard and some slips of paper of the right size or even play with pen or pencil on graph paper (but the problem there is that it will be hard to undo moves when you make a mistake!). The rules of DOMINEERING are simple. Two players alternately place dominoes on the chessboard. A domino can only be placed so that it covers two adjacent squares. One player, Louise, places her dominoes so that they cover vertically adjacent squares. The other player, Richard, places his dominoes so that they cover horizontally adjacent squares. The game ends when one of the players is unable to place a domino, and that player then loses. Here is a sample game on a 4×6 board with Louise moving first:



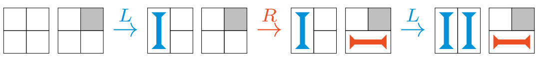
Since Louise placed the last domino, she has won.

Exercise 0.1. Stop reading! Find a friend and play some games of DOMINEERING. A game on a full chessboard can last a while so you might want to play on a 6×6 square to start with.

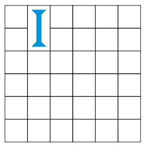
If you did the exercise, then you probably made some observations and learned a few tactical tricks in DOMINEERING. One observation is that after a number of dominoes have been placed the board *falls apart* into disconnected regions of empty squares. When you make a move you need to decide what region to play in and how. Suppose that you are the vertical player and that there are two regions of the form:

 and 

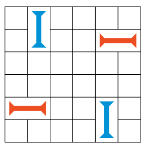
Obviously you could move in either region. However, if you move in the hook-shaped region, then your opponent will move in the square. You will have no more moves left so you will lose. If instead you move in the square, then your opponent's only remaining move is in the hook. Now you still have a move in the square to make, and so your opponent will lose. If you are L and your opponent is R, play should proceed as



This is also why an opening move such as



is good since it *reserves* the two squares in the upper left for you later. In fact, if you play seriously for a while it is quite possible that the board after the first four moves will look something like



Simply put, the aim of combinatorial game theory is to understand in a more detailed way the principles underlying the sort of observations that we have just made about DOMINEERING. We will learn about games in general and how to understand them but, as a bonus, how to play them well!

0.1 Basic Terminology

In this section we will provide an informal introduction to some of the basic concepts and terminology that will be used in this book and a description of how combinatorial games differ from some other types of games.

Combinatorial games

In a *combinatorial game* there are two players who take turns moving alternately. Play continues until the player whose turn it is to move has no legal moves available. No chance devices such as dice, spinners, or card deals are involved, and each player is aware of all the details of the game position (or game state) at all times. The rules of each game we study will ensure that it must end after a finite sequence of moves, and the winner is often determined on the basis of who made the last move. In *normal play* the last player to move wins. In *misère game play* the last player loses.

In fact, combinatorial game theory can be used to analyze some games that do not quite fit the above description. For instance, in DOTS & BOXES, players may make two (or more) moves in a row. Most CHECKERS positions are *loopy* and can lead to infinitely long sequences of moves. In GO and CHESS the last mover does not determine the winner. Nonetheless, combinatorial game theory has been applied to analyze positions in each of these games.

By contrast, the classical mathematical theory of games is concerned with *economic games.* In such games the players often play simultaneously and the outcome is determined by a payoff matrix. Each player's objective is to guarantee the best possible payoff against any strategy of the opponent. For a taste of economic game theory, see Problem 5.

The challenge in analyzing economic games stems from simultaneous decisions: each player must decide on a move without knowing the move choice(s) of her opponent(s). The challenge of combinatorial games stems from the sheer quantity of possible move sequences available from a given position.

Combinatorial game theory is most straightforward when we restrict our attention to *short games.* In the play of a short game, a position may never be repeated, and there are only a finite number of other positions that can be reached. We implicitly (and sometimes explicitly) assume all games are short in this text.

Introducing the players

The two players of a combinatorial game are traditionally called *Left* (or just L) and *Right* (R). Various conventional rules will help you to recognize who is playing, even without a program:

Left	**Right**
Louise	Richard
Positive	Negative
bLack	White
bLue	Red
Vertical	Horizontal
Female	Male
Green	
Gray	

Alice and *Bob* will also make an appearance when the first player is important (Alice moves first). To help remember all these conventions, note that despite the fact that they were introduced as long ago as the early 1980s in *Winning Ways* (*WW*) [BCG01], the chosen dichotomies reflect a relatively modern "politically correct" viewpoint.

Often we will need a neutral color, particularly in pen and paper games or games involving pieces. If the game is between blue and red then this neutral color is green (because green is good for everyone!), while if it is between black and white then the neutral color is gray (because gray is neither black nor white!). Because this book is printed in color, games traditionally played in black and white (and gray) are presented in color instead. That is,

$$\begin{aligned} \text{black} &= \text{blue}, \\ \text{white} &= \text{red}, \\ \text{gray} &= \text{green}. \end{aligned}$$

Options

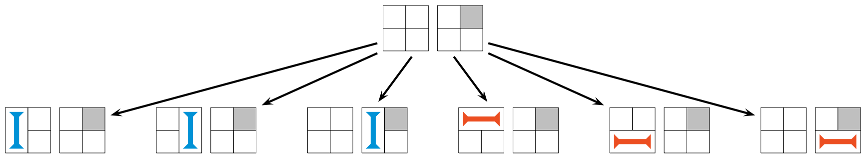
If a position in a combinatorial game is given and it happens to be Left's turn to move, she will have the opportunity to choose from a certain set of moves determined by the rules of the game. For instance in DOMINEERING, where Left plays the vertical dominoes, she may place such a domino on any pair of vertically adjacent empty squares. The positions that arise from exercising these choices are called the *left options* of the original position. Similarly, the *right options* of a position are those that can arise after a move made by Right. The *options* of a position are simply the elements of the union of these two sets.

We can draw a *game tree* of a position (as a directed tree) by the following procedure.

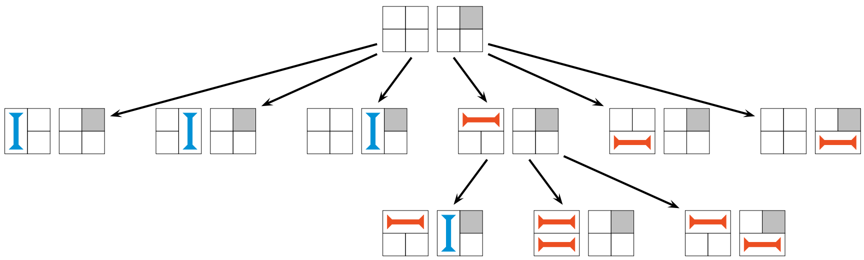
- Create a node for the original position, and draw nodes for each of its options, placing them below the first node. Then draw a directed edge from the top node to its options.
- For each option, again draw nodes for each of *its* options, placing them below and drawing a directed edge to these new nodes.
- Repeat with any subposition that still has unexpanded options.

The nodes of the game tree correspond to the *followers* of the original position and are *all* the positions that result from any sequence of moves. The followers include the original position (the empty sequence of moves!) and also sequences in which players may get several moves in row. It is also possible to have a position appear in many places of the tree. The *game graph* is obtained by merging all the nodes that correspond to a single position, but the game tree is more important for induction purposes.

As a visual aid, our game trees will have the left options appearing below and to the left of the game and right options below and to the right. Often, induction will be based on the options, and we will draw a *partial game tree* consisting only of the original position and its options:

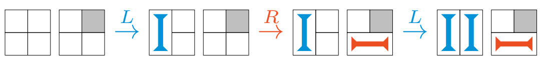


Occasionally, we will include some other interesting followers:

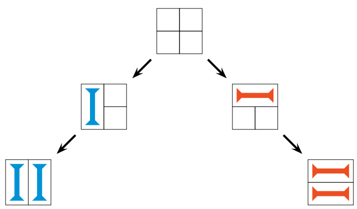


It may seem odd that we are showing two consecutive right moves in a game tree, but much of the theory of combinatorial games is based on analyzing situations where games *decompose* into several subgames. It may well be the case that in some of the subgames of such a decomposition, the players do not alternate moves.

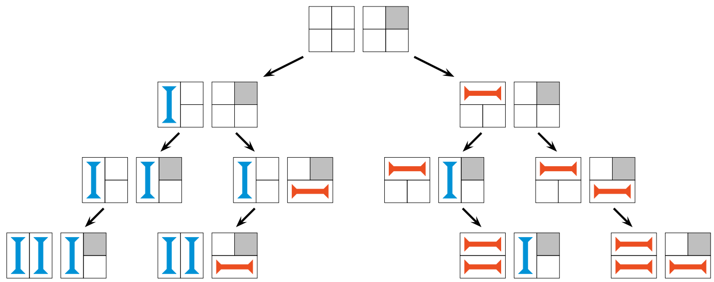
We saw this already in the DOMINEERING "square and hook" example. Left, if she wants to win, winds up making two moves in a row in the square:



Thus, we show the game tree for a square with Left and/or Right moving twice in a row:



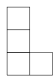
As we will see later in Chapter 4, *dominated options* are often omitted from the game tree, when an option shown is at least as good:



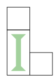
In some games the left options and the right options of a position are always the same. Such games are called *impartial*. The study of impartial combinatorial games is the oldest part of combinatorial game theory and dates back to the early twentieth century. On the other hand, the more general study of non-impartial games was pioneered by John Conway in *On Numbers and Games* (*ONAG*) [Con01] and by Elwyn Berlekamp, John Conway, and Richard Guy in *WW* [BCG01]. Since "non-impartial" hardly trips off the tongue, and "partial" has a rather ambiguous interpretation, it has become commonplace to refer to non-impartial games as *partizan games*.

To illustrate the difference between these concepts, consider a variation of DOMINEERING called CRAM. CRAM is just like DOMINEERING except that each player can play a domino in either orientation. Thus, it becomes an impartial game since there is now no distinction between legal moves for one player and legal moves for the other.

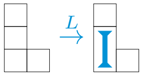
Let's look at a position in which there are only four remaining vacant squares in the shape of an L:



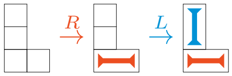
In CRAM the next player to play can force a win by playing a vertical domino at the bottom of the vertical strip, leaving



which contains only two non-adjacent empty squares and hence allows no further moves. In DOMINEERING if Left (playing vertically) is the next player, she can win in exactly this way. However, if Right is the next player his only legal move is to cover the two horizontally adjacent squares, which still leaves a move available to Left. So (assuming solid play) Left will win regardless of who plays first:



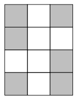
or



Much of the theory that we will discuss is devoted to finding methods to determine who will win a combinatorial game assuming sensible play by both sides. In fact, the eventual loser has no really *sensible* play[1] so a *winning strategy* in a combinatorial game is one that will guarantee a win for the player employing it no matter how his or her opponent chooses to play. Of course, such a strategy is allowed to take into account the choices actually made by the opponent — to demand a uniform strategy would be far too restrictive!

Problems

1. Consider the position



(a) Draw the complete game trees for both CRAM and DOMINEERING. The leaves (bottoms) of the tree should all be positions in which neither player can move. If two left (or right) options are symmetrically identical, you may omit one.

(b) In the position above, who wins at DOMINEERING if Vertical plays first? Who wins if Horizontal plays first? Who wins at CRAM?

2. Suppose that you play DOMINEERING (or CRAM) on *two* 8×8 chessboards. At your turn you can move on either chessboard (but not both!). Show that the second player can win.

3. Take the ace through five of one suit from a deck of cards and place them face up on the table. Play a game with these as follows. Players alternately pick a card and add it to the right-hand end of a row. If the row ever contains a sequence of three cards in increasing order of rank (ace is low), or in decreasing order of rank, then the game ends and the player who formed that sequence is the winner. Note that the sequence

[1]Unless he has some ulterior motive not directly related to the game such as trying to make it last as long as possible so that the bar closes before he has to buy the next round of drinks.

need not be consecutive either in position or value, so, for instance, if the play goes $4, 5, 2, 1$ then the $4, 2, 1$ is a decreasing sequence.

(a) Show that this is a proper combinatorial game (the main issue is to show that draws are impossible).

(b) Show that the first player can always win.

4. Start with a heap of counters. As a move from a heap of n counters, you may either:

- assuming n is not a power of 2, remove the largest power of 2 less than n; or
- assuming n is even, remove half the counters.

Under normal play, who wins? How about misère play?

5. The goal of this problem is to give the reader a taste of what is *not* covered in this book. Two players play a 2×2 *zero-sum matrix game.* (*Zero sum* means that whatever one person loses, the other gains.) The players are shown a 2×2 matrix of positive numbers. Player A chooses a row of the matrix, and player B simultaneously chooses a column. Their choice determines one matrix entry, that being the number of dollars B must pay A. For example, suppose the matrix is

$$\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}.$$

If player A chooses the first row with probability $\frac{1}{4}$, then no matter what player B's strategy is, player A is *guaranteed* to get an average of \$2.50. If, on the other hand, player B chooses the columns with 50-50 odds, then no matter what player A does, player B is *guaranteed* to have to pay an average of \$2.50. Further, neither player can guarantee a better outcome, and so B should pay player A the fair price of \$2.50 to play this game.

In general, if the entries of the matrix game are

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

as a function of a, b, c, and d, what is the fair price that B should pay A to play? (Your answer will have several cases.)

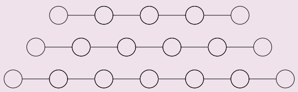
Preparation for Chapter 1

Prep Problem 1.1. Play DOTS & BOXES with a friend or classmate. The rules are found on page 307 of Appendix D. You should start with a 5×6 grid of dots. You should end up with a 4×5 grid of 20 boxes, so the game might end in a tie.

When playing a game for the first time, feel free to move quickly to familiarize yourself with the rules and to get a sense for what can happen in the game.

After a few games of DOTS & BOXES, write a few sentences describing any observations you have made about the game. Perhaps you found a juncture in the game when the nature of play changes? Did you have a strategy? (It need not be a good strategy.)

Prep Problem 1.2. Play SNORT with a friend or classmate. The rules are found on page 313 of Appendix D. (Note that if the Winner is not specified in a ruleset, you should assume normal play, that the last legal move wins.) You should play on paths of various lengths: for instance,



Jot down any observations you have about the game, and then try playing COL on the same initial positions.

Prep Problem 1.3. Play CLOBBER with a friend or classmate. The rules are found on page 305 of Appendix D. You should start with a 5×6 grid of boxes:

Jot down any observation you have about the game.

Prep Problem 1.4. Play NIM (rules on page 311) with a friend or classmate. Begin with the three heap position with heaps of sizes 3, 5, and 7.

To the instructor: While DOTS & BOXES is a popular topic among students, it also takes quite a bit of time to appreciate. View the topic as optional. If you do cover it, allow time for students to play practice games. Another option is to cover it later in the term before a holiday break.

Chapter 1
Basic Techniques

If an enemy is annoying you by playing well,
consider adopting his strategy.

Chinese proverb

There are some players who seem to be able to play a game well immediately after learning the rules. Such gamesters have a number of tricks up their sleeves that work well in many games without much need for further analysis. In this chapter we will teach you some of these tricks or, to use a less emotive word, *heuristics*.

Of course, the most interesting games are those to which none of the heuristics apply directly, but knowing them is still an important part of getting started with the analysis of more complex games. Often, you will have the opportunity to consider moves that lead to simple positions in which one or more of the heuristics apply. Those positions are then easily understood, and the moves can accordingly be taken or discarded.

1.1 Greedy

The simplest of the heuristic rules or strategies is called the *greedy strategy*. A player who is playing a greedy strategy grabs as much as possible whenever possible. Games that can be won by playing greedily are not terribly interesting at all — but most games have some aspects of greedy play in them. For instance, in CHESS it is almost always correct to capture your opponent's queen with a piece of lesser value (taking a greedy view of "getting as much material advantage as possible"), but not if doing so allows your opponent to capture your queen, or extra material, or especially not if it sets up a checkmate for the opponent. Similarly, the basic strategy for drawing in TIC TAC TOE is a greedy one based on the idea "always threaten to make at least one line, or block any threat of your opponent."

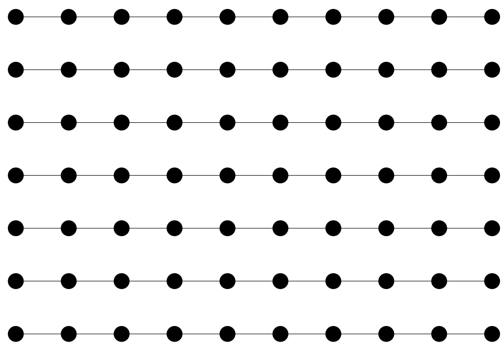
Definition 1.1. A player following a *greedy strategy* always chooses the move that maximizes or minimizes some quantity related to the game position after the move has been made.

Naturally, the quantity on which a greedy strategy is based should be easy enough to calculate that it does not take too long to figure out a move. If players accumulate a *score* as they play (where the winner is the one who finishes with the higher score), then that score is a natural quantity to try to maximize at each turn.

Does a greedy strategy always work? Of course not, or you wouldn't have a book in front of you to read. But in some very simple games it does. In the game GRAB THE SMARTIES[1] each player can take at his or her move any number of Smarties from the box, provided that they are all the same color. Assuming that each player wants to collect as many Smarties as possible, the greedy strategy is ideal for this sort of game. You just grab all the Smarties of some color, and the color you choose is the one for which your grab will be biggest.

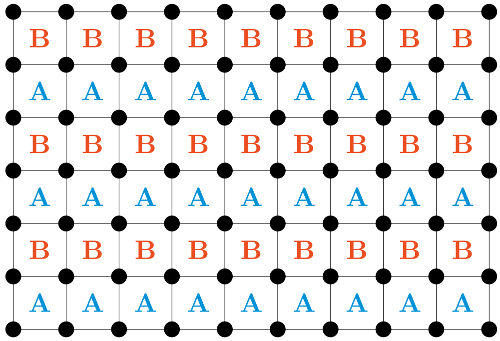
Sometimes though, a little subtlety goes a long way.

Example 1.2. Below is the board after the first moves in a very boring game of DOTS & BOXES:

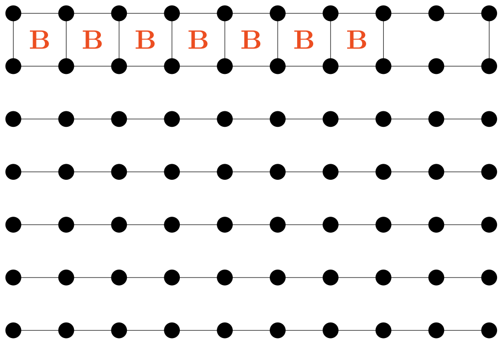


Suppose that it is now Alice's turn. No matter where Alice moves, Bob can take all the squares in that row. If he does so, he then has to move in another row, and Alice can take all the squares in this row. They trade off in this way until they both have 27 boxes and the game is tied:

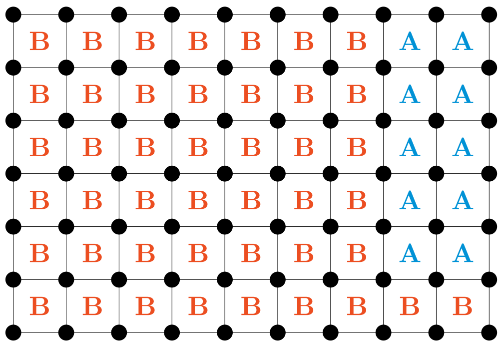
[1] An American player might play the less tasty variant, GRAB THE M&M'S.



But Bob is being too greedy. Instead of taking all the squares in the row that Alice opens up, he should take all but two of them, then make the *double-dealing move,* which gives away the last two boxes to Alice. For example, if Alice moves somewhere toward the left-hand end of the first row, Bob replies with



Alice now has a problem. Regardless of whether she takes the two boxes that Bob has left for her, she still has to move first in another row. So she might as well take the last two in a *double-cross* (since otherwise Bob will get them on his next turn), but she then has to give seven boxes of some other row to Bob in his next turn. By repeating this strategy for each but the last row, where he takes all the boxes, Bob finishes with $5 \times 7 + 9 = 44$ boxes while Alice gets only $5 \times 2 = 10$ boxes:



Again, a move that makes two boxes with one stroke is called a *double-cross.* (A person who makes such a play might feel double-crossed for having to reply to a *double-dealing move.*)

It is not a bad idea to use a greedy strategy on your first attempt at playing a new game, particularly against an expert. It is easy to play, so you won't waste time trying to figure out good moves when you don't know enough about the game to do so. And when the expert makes moves that refute your strategy (i.e., exposes the traps hidden in the greedy strategy), then you can begin to understand the subtleties of the game.

Exercise 1.3. Now that you have learned a bit of strategy, play DOTS & BOXES against a friend. (If you did the prep problems for this chapter, you already played once. Now, you may be able to play better.)

1.2 Symmetry

A famous CHESS wager goes as follows: An unknown CHESS player, Jane Pawnpusher, offers to play two games, playing the white pieces against Garry Kasparov and black against Anatoly Karpov simultaneously. She wagers $1 million dollars that she can win or draw against one of them. Curiously, she can win the wager without knowing much about CHESS. How?

What she does is simply wait for Karpov to make a move (white moves first in CHESS), and whatever Karpov does, she makes the same move as her first move against Kasparov. Once Kasparov replies, she plays Kasparov's reply against Karpov. If Kasparov beats her, she will beat Karpov the same way.[2] A strategy that maintains a simple symmetry like this is called *Tweedledum-Tweedledee* or *copycat.*

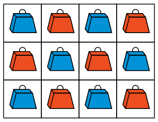
Example 1.4. The Tweedledum-Tweedledee strategy is effective in two-heap NIM. If the two heaps are of the same size, then you should invite your opponent to move first. She must choose a heap and remove some counters. You choose the other heap and take away the same number of counters leaving two equal sized heaps again. On the other hand, if the game begins with two heaps of different sizes, you should rush to make the first move, taking just enough counters from the larger heap to make them equal. Thereafter, you adopt the Tweedledum-Tweedledee approach.

Symmetry is an intuitively obvious strategy. Whenever your opponent does something on one part of the board, you should mimic this move in another part. Deciding how this mimicry should happen is the key. To be played

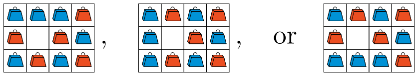
[2] Readers familiar with cryptography may observe Jane is making a man-in-the-middle attack.

successfully, you should not leave a move open to your opponent that allows him to eliminate your mimicking move.

Example 1.5. If Blue moves from the 3×4 CLOBBER game



to any of the three positions

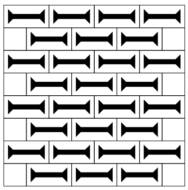


then Red can play the remainder of the game using a 180 degree symmetry strategy. This establishes that each of these three moves for Blue was a poor choice on her first turn. In fact, from this position it happens to be the case that she simply has no good first moves, but the rest of her initial moves cannot be ruled out so easily due to symmetry.

Sometimes, symmetry can exist that is not apparent in the raw description of a game.

Example 1.6. Two players take turns putting checkers down on a checkerboard. One player plays blue, one plays red. A player who completes a 2×2 square with four checkers of one color wins.

This game should end in a draw. First, imagine that most of the checkerboard is tiled with dominoes using a brickwork pattern:



If your opponent plays a checker in a domino, you respond in the same domino. If you cannot (because you move first, or the domino is already filled, or your opponent fails to play in a domino), play randomly. Since every 2×2 square contains one complete domino, your opponent cannot win. Therefore, both players can force at least a draw, and neither one can force a win.

Exercise 1.7. Two players play $m \times n$ CRAM.

(a) If m and n are even, who should win? The first player or the second player? Explain your answer.

(b) If m is even and n is odd, who should win? Explain.

(When m and n are odd, the game remains interesting.)

1.3 Parity

Parity is a critical concept in understanding and analyzing combinatorial games. A number's *parity* is whether the number is odd or even. In lots of games, only the parity of a certain quantity is relevant — the trick is to figure out just what quantity! With the normal play convention that the last player with a legal move wins, it is always the objective of the first player to play to ensure that the game lasts an odd number of moves, while the original second player is trying to ensure that it lasts an even number of moves.

This is part of the reason why symmetry as we mentioned earlier is also important — it allows the second player (typically) to think of moves as being blocked out in pairs, ensuring that he has a response to any move his opponent might make.

The simplest game for which parity is important is called SHE LOVES ME SHE LOVES ME NOT. This game is played with a single daisy. The players alternately remove exactly one petal from the daisy and the last player to remove a petal wins. Obviously, all that matters is the original parity of the number of petals on the daisy. If it is odd then the first player will win; if it is even then the second player will win.

More usually SHE LOVES ME SHE LOVES ME NOT is delivered in some sort of disguise.

Example 1.8. Take a heap of 29 counters. A move is to choose a heap (at the start there is only one) and split it into two non-empty heaps. Who wins?

Imagine the counters arranged in a line. A move effectively is to put a bar between two counters. This corresponds to splitting a heap into two: the counters to the left and those to the right up to the next bar or end of the row. There are exactly 28 moves in the game. The game played with a heap of n counters has exactly $n-1$ moves! The winner is the first player if n is even and the second player if n is odd.

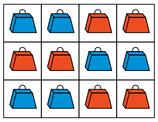
Exercise 1.9. A chocolate bar is scored into smaller squares or rectangles. (Lindt's Swiss Classic, for example, is 5×6.) Players take turns picking up one piece (initially the whole bar), breaking the piece into two along a scored line, and setting the pieces back down. The player who moves last wins. Our goal is to determine *all* winning moves.

1.4 Give Them Enough Rope!

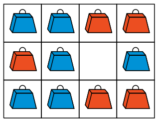
The previous strategies are all explicit, and when they work, you can win the game. This section is about confounding your opponent in order to gain time for analysis.

If you are in a losing position, it pays to follow the *Enough Rope Principle:* Make the position as complicated as you can with your next move.[3] Hopefully, your opponent will tie himself up in knots while trying to analyze the situation.

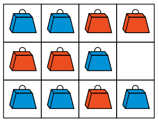
For example, suppose you are Blue and are about to move from the following CLOBBER position:



If you are more astute than the authors, you could conclude that you have no winning moves. However, you should probably not throw in the towel just yet. But you also should not make any moves for which your opponent has a simple strategy for winning. If your opponent has read this chapter, you should avoid capturing an edge piece with your center piece, for then Red can play a rotational symmetry strategy. However, there are several losing responses from either of the positions



or



and so these moves, while losing, are reasonable.

The Enough Rope Principle has other implications as well. If you are confused about how best to play, do not simplify the position to the point where your opponent will not be confused, especially if you are the better player.

The converse applies as well. If you are winning, play moves that simplify. Do not give you opponent opportunities to complicate the position, lest you be hoist by your own petard.

Don't give them any rope

This is contrary to the advice in the rest of the section. If you do not know that you are losing the game and you are playing against someone of equal or less game-playing ability, then a very good strategy is to move so as to restrict the number of options that your opponent has and increase the number of your own options. This is a heuristic that is often employed in first attempts to produce programs that will play games reasonably well. This has been used in AMAZONS, CONNECT-4, and OTHELLO.

1.5 Strategy Stealing

Strategy stealing is a technique whereby one player steals another player's strategy. Why would you want to steal a strategy? Let's see . . . .

[3] At least one of the authors feels compelled to add, *except if you are playing against a small child.*

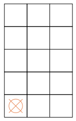
Two-move chess

Players play by the ordinary rules of chess, but each player plays two consecutive regular chess moves on each turn (this example appears in [SA03]).

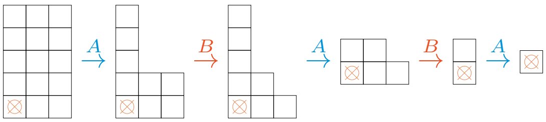
White, who moves first, can under perfect play win or draw. For if Black had a winning strategy, White could steal it by playing a Knight out and then back again. Black is now faced with making the initial foray on the board with the roles of Black and White reversed.

Chomp

The usual starting position of a game of CHOMP consists of a rectangle with one poison square in the lower-left corner:



A move in CHOMP is to choose a square and to remove it and all other squares above or to the right of it. A game between players Alice and Bob might progress as follows:



And Bob loses for he must take the poison square.

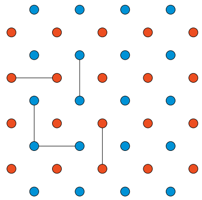
Theorem 1.10. CHOMP, *when played on a rectangular board larger than* 1×1, *is a win for the first player.*

Proof: Suppose that the first player chomps *only* the upper-right square of the board. If this move wins, then it is a first-player win. If, on the other hand, this move loses, then the second player has a winning response of chomping all squares above or to the right of some square, x. But move x was available to the first player on move one, and it removes the upper-right square, so the first player has move x as a winning first move. □

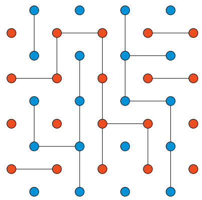
This is a *non-constructive proof* in that the proof gives no information about what the winning move is. The proof can be rephrased as a *guru argument,* echoing the CHESS wager of Section 1.2.

Bridg-it

The game of BRIDG-IT is played on two offset grids of blue and red dots. Here is a position after five moves (Blue played first):

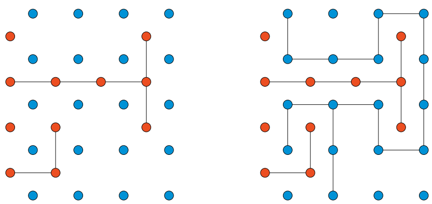


The players, Blue and Red, alternate drawing horizontal or vertical lines joining adjacent dots of the player's chosen color. Blue wins by connecting any dot in the top row to any dot in the bottom row by a path. Red is trying to connect the left side to the right side. In the following position, Blue has won, with a path near the right side of the board:



Lemma 1.11. BRIDG-IT *cannot end in a draw.*

Sketch of proof: The game is unaffected if we consider the top and bottom rows of blue dots as connected. Suppose the game has ended, and neither player has won. Let S be the set of nodes that Red can reach from the left side of the board. Then, starting from the upper-left blue dot, Blue can go from the top to the bottom edge by following the boundary of the set S. As an example, set S consists of the red dots below, and the blue path following the boundary is shown on the right:



All the edges connecting blue dots must be present, for otherwise set S could be extended. □

Theorem 1.12. *The first player wins at* BRIDG-IT, *where the starting board is an* $n \times (n+1)$ *grid of red dots overlapping an* $(n+1) \times n$ *grid of blue dots.*

Proof: Note that the board is symmetric when reflected about the main diagonal. If the second player has a winning strategy, the first player can adopt it. In particular, before her first move, the first player pretends that some random invisible move x has been made by the opponent and then responds as the second player would have. If the opponent's n^{th} move is move x, then the first player pretends that the opponent actually played the n^{th} move at some other location x'. Continuing in this fashion, the first player has stolen the winning second-player strategy, with the only difference that the opponent always has one fewer lines on the board. This missing move can be no worse for the first player, and so she will win. □

There are explicit winning strategies for BRIDG-IT, avoiding the need for a non-constructive strategy-stealing argument. So, not only can you find out you *should* win, but by utilizing a bit of elementary graph theory, you can also find out *how* to win! See, for example, [BCG01, volume 3, pp. 744–746] or [Wes01, pp. 73–74].

1.6 Change the Game!

Sometimes a game is just another game in disguise. In that case one view can be more, or less, intuitive than the other.

Example 1.13. In 3-TO-15, there are nine cards, face up, labeled with the digits $\{1, 2, 3, \ldots, 9\}$. Players take turns selecting one card from the remaining cards. The first player who has three cards adding up to 15 wins.

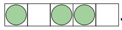
This game should end in a draw. Surprisingly, this is simply TIC TAC TOE in disguise! To see this, construct a *magic square* where each row, column, and diagonal add up to 15:

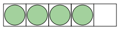
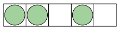
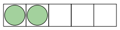
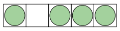
4	9	2
3	5	7
8	1	6

You can confirm that three numbers add up to 15 if and only if they are in the same TIC TAC TOE line. Thus, you can treat a play of 3-TO-15 as play of TIC TAC TOE. Suppose that you are moving first. Choose your TIC TAC TOE move, note the number on the corresponding square, and select the corresponding card. When your opponent replies by choosing another card, mark the TIC TAC TOE board appropriately, choose your TIC TAC TOE response, and again take the corresponding card. Proceed in this fashion until the game is over. So if you can play TIC TAC TOE, you can play 3-TO-15 just as well.

Exercise 1.14. Play 3-TO-15 against a friend. As you and your friend move, mark the magic square TIC TAC TOE board with Xs and Os to convince yourself that the games really are the same.

Example 1.15. COUNTERS is played with tokens on a $1 \times n$ strip of squares. Players can put down a counter on an empty square or move a counter leftward to the next empty square. In the latter case all the counters that it jumps over (if any) are removed. POWER 2 is a game whose position is a positive integer n and a move is to subtract a power of 2 from n provided that the result is non-negative.

These games are the same! For example, consider the position . Change empty spaces to 1's and counters to 0's and the position is 01001, which is the binary expansion for 9 (with a leading zero). The possible moves (and their corresponding numbers) are

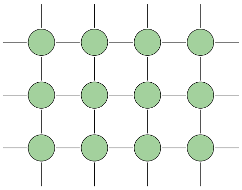
 $= 00001 = 1 = 9 - 8$,
 $= 00101 = 5 = 9 - 4$,
 $= 00111 = 7 = 9 - 2$, and
 $= 01000 = 8 = 9 - 1$.

Problem 10 of Chapter 7 asks you to solve POWER 2.

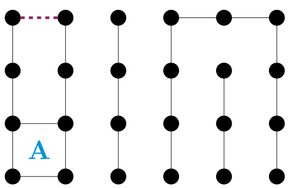
1.7 Case Study: Long Chains in Dots & Boxes

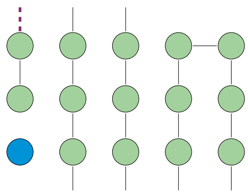
We already observed in Example 1.2 on page 12 that the first player to play on a long chain in a DOTS & BOXES game typically loses. In this section, we will investigate how that can help a player win against any first grader.

First, consider a dual form of DOTS & BOXES called STRINGS & COINS. Here is a typical starting position:



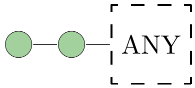
A move in STRINGS & COINS consists of cutting a string. If a player severs the last of the four strings attached to a coin, the player gets to pocket the coin and must move again. This game is the same as DOTS & BOXES but disguised: a coin is a box, and cutting a string corresponds to drawing a line between two boxes. For example, here is a DOTS & BOXES position and its dual STRINGS & COINS position:

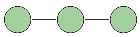




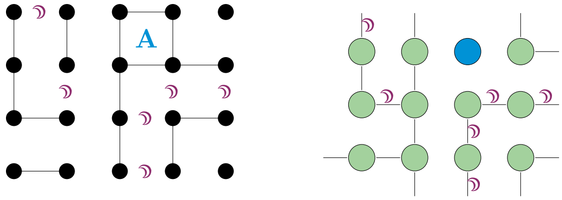
Drawing the dotted line in DOTS & BOXES corresponds to cutting the dotted string in STRINGS & COINS. We will investigate STRINGS & COINS positions, since people tend to find that important properties of the positions (such as long chains) are easier to visualize and identify in this game than in DOTS & BOXES.

Positions of the following form are termed *loony:*

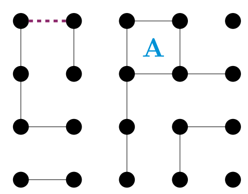
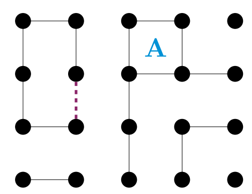
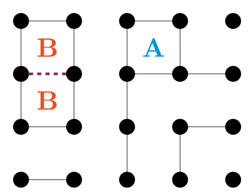
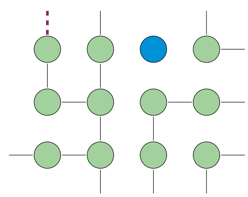
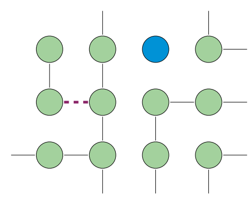
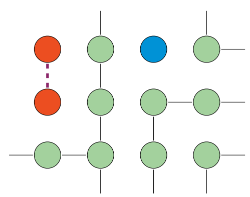


is loony except  is not loony.

The hidden portion of the position (in the box marked ANY) can be any position except a single coin. The defining characteristic shared by all loony positions is that the next player to move has a choice of whether or not to make a double-dealing move. A *loony move*, denoted by a ☽, is any move to a loony position. All loony moves are labeled in the following DOTS & BOXES and equivalent STRINGS & COINS positions:



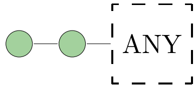
To summarize, if Bob makes a loony move (a move to a loony position), Alice may (or may not) reply with a double-dealing move. From there, Bob might as well double-cross before moving elsewhere.

Loony move	**Double-dealing move**	**Double-cross**
		
		

(Bob must move again after the double-cross.)

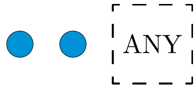
Theorem 1.16. *Under optimal play from a loony position, the player to move next can get at least half the remaining coins.*

Proof: Suppose that player Alice is to move next from the loony position

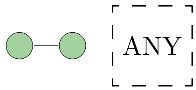


Consider playing only on the hidden position in the box marked ANY (without the two extra coins). Suppose that the player moving next from ANY can guarantee pocketing n coins. In the full position, Alice has at least two choices:

- Pocket the two coins (making two cuts) and move on ANY, pocketing n more coins for a total of $n+2$:



- Sacrifice the two coins, cutting off the pair in one move. Whether or not the opponent chooses to pick up the two coins, he must move next on ANY, and so the most he can pocket is $n+2$ coins:

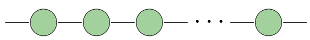


Thus, Alice can collect $n+2$ coins, or all but $n+2$ coins. One of these two numbers is at least half the total number of coins! □

In practice, the player about to move often wins decisively, especially if there are very long chains.

Hence, from most positions, a ☽ move (i.e., a move to a ☽ position) is a losing move and might as well be illegal when making a first pass at understanding a position.

A *long chain* consists of $k \geq 3$ coins and *exactly* $k+1$ strings connected in a line:



Notice that any move on a long chain is ☽.

Exercise 1.17. Find the non-☽ move(s) on a (short) chain of length 2:

Exercise 1.18. Here are two separate STRINGS & COINS positions. Alice is about to play in each game:

(a) Both positions are loony. Explain why.

(b) In one of the positions, Alice should make a double-dealing move. Which one? Why?

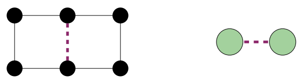
(c) Estimate the score of a well-played game from each of the two positions. (Alice should be able to win either game.)

So, it is crucial to know whose move it is if only long chains remain, for in such a position all moves are loony and Theorem 1.16 tells us that the player about to move will likely lose. To this end, consider a position with only long chains. We distinguish a move (drawing a line or cutting a string) from a turn, which may consist of several moves.

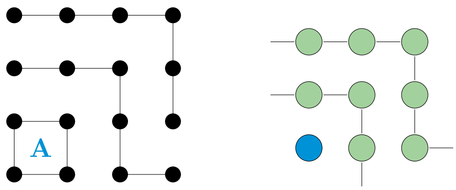
Define

M^- = number of moves played so far;
M^+ = number of moves remaining to be played;
M = $M^+ + M^-$ = possible moves from the start position;
T = number of turn transfers so far;
B^- = number of boxes (or coins) taken already;
B^+ = number of boxes (or coins) left to be taken;
B = $B^+ + B^-$ = total number of boxes (or coins) in the start position;
C = number of long chains;
D = number of double-crosses so far.

Recall that double-crosses are single moves that take two coins in one cut (or complete two boxes in one stroke):



We can compute the above quantities for the following position:



$$\begin{aligned}
M^- &= 14,\\
M^+ &= 10,\\
M &= 24,\\
T &= 13,\\
B^- &= 1,\\
B^+ &= 8,\\
B &= 9,\\
C &= 2,\\
D &= 0.
\end{aligned}$$

(If there were two adjacent boxes taken by the same player, we would not know if $D = 0$ or $D = 1$ without having watched the game in progress.)

We will now proceed to set up equations describing (as best we can) the state of affairs when we are down to just long chains.

- Since every long chain has one more string than coin, we know that

$$M^+ = C + B^+.$$

- Since every move either completes a turn, completes a box, or completes two boxes,

$$M^- = T + B^- - D.$$

Adding these equations, we conclude that

$$M = C + T + B - D.$$

Whose turn it is depends only on whether T is even or odd; M and B are fixed at the start of the game, so whose turn it is is determined by the number of long chains and the number of double crosses. We have all but proved the following:

Theorem 1.19. *If a* STRINGS & COINS *(or* DOTS & BOXES*) position is reduced to just long chains, player P can earn most of the remaining boxes, where*

$$P \equiv M + C + B + D \pmod 2,$$

the first player to move is player $P = 1$, and her opponent is player $P = 2$ (or, if you like, $P = 0$).

Proof: By the discussion preceding the theorem,

$$M = C + T + B - D.$$

If all that remains are long chains, whoever is on move (i.e., about to move) must make a loony move, which by Theorem 1.16 guarantees that the last player can take at least half the remaining coins. If you are player 1, say, then your opponent is on move if an odd number of turns have gone by since the start of the game; i.e., T is odd. Note that T is odd if and only if $M-C-B+D$ is odd; i.e., if and only if $P \equiv M+C+B+D \pmod 2$. (Replace odd by even for $P=2$.) □

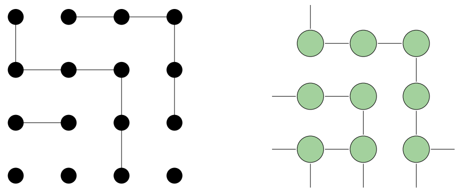
In a particular game, viewed from a particular player's perspective, P, M, and B are all constant, and D is nearly always 0 (until someone makes a loony move.) So, the parity of T depends only on C, a quantity that depends on the actual moves made by the players.

In summary, when you sit down to a game of DOTS & BOXES, count

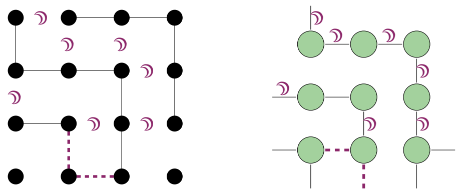
$$P+M+B,$$

where P is your player number. You seek to ensure that the parity of C, the number of long chains, matches this quantity.[4] That is, you want to arrange that $C \equiv P+M+B \pmod 2$. If you can play to make the parity of the long chains come out in your favor, you will usually win.

An example is in order. Alice played first against Bob, and they reach the following position with Alice to play:



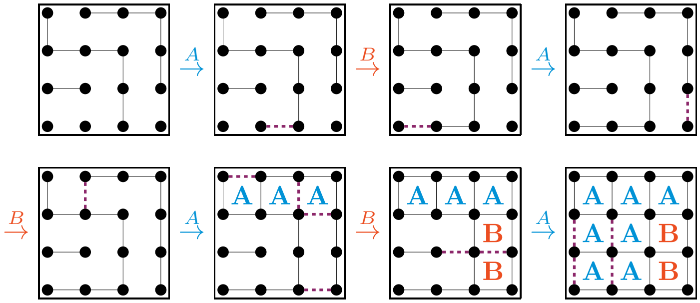
At the start of the game, Alice computed $P+M+B = 1+24+9$, an even quantity, and therefore knows that she wishes for an even number of long chains. Having identified all loony moves, she knows that the chain going around the upper and right sides will end in one long chain *unless* someone makes a loony (losing) move. So, she hopes that the lower-left portion of the board ends in a long chain. Two moves will guarantee that end, those marked below with dashed lines:



Of the two moves, Alice prefers the horizontal move, since that lengthens the chain; because she expects to win most of the long chains, longer chains favor

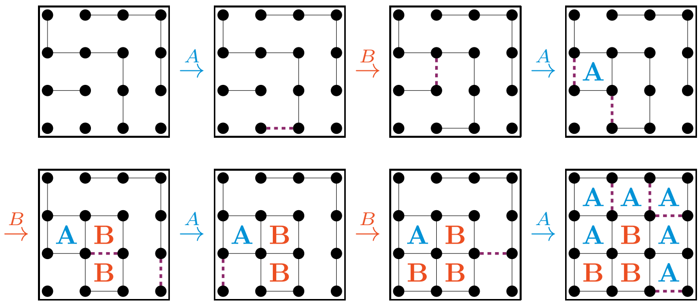
[4]In rectangular DOTS & BOXES boards, the number of dots is $1+M+B \pmod 2$, and some players prefer to count the dots. This is the view taken in [Ber00].

her. If Bob is a beginner, a typical game might proceed as



with Alice winning 7 to 2.

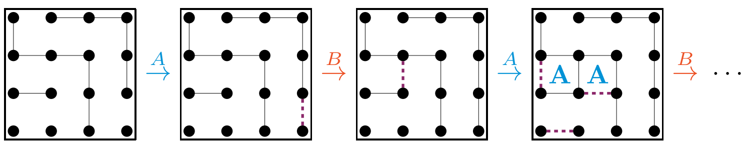
A more sophisticated Bob might recognize that he has lost the game of long chains and might try to finagle a win by playing loony moves earlier. This has the advantage of giving Alice fewer points for her long chains. A sample game between sophisticated players might go



Not only does Bob lose by only 6 to 3, but Bob might win if Alice hastily takes all three boxes in the first long chain!

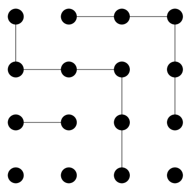
Exercise 1.20. What is the final score if Alice takes all three boxes instead of her first double-dealing move in the last game? Assume that both players play their best thereafter.

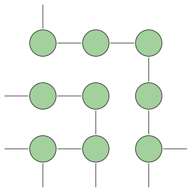
Suppose that Alice fails to make a proper first move. Bob can then steal control by sacrificing two boxes (without making a loony move), breaking up the second long chain. For example, play might proceed as



In this game, Alice should get the lower-left four boxes, but Bob will get the entire upper-right chain, winning 5 to 4.

Lastly, note that nowhere in the discussion leading up to or in the proof of Theorem 1.19 did we use any information about what the start position was. Consequently, if you come into a game already in play, you can treat the current position as the start position! In our example game between Alice and Bob repeated below, since M, the number of moves available from this start position, is fourteen and B, the number of boxes still available, is nine, the player on move (who we dub player 1 from this start position) wants an even number of long chains:

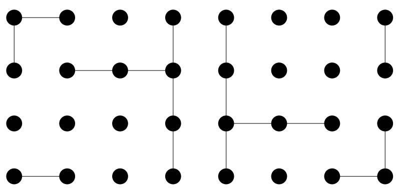


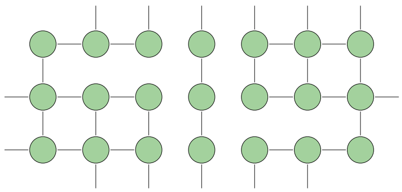


Warning: Not all games reach an endgame consisting of long chains. There are other positions in which all moves are loony. See, for example, Problem 18. These sorts of positions come up more often the larger the board size.

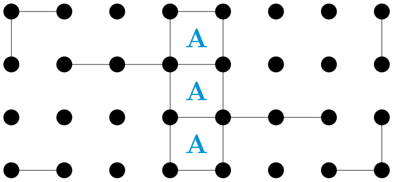
Enough Rope Principle revisited

In the following DOTS & BOXES (or the equivalent STRINGS & COINS) position, Bob has stumbled into Alice's trap, and Alice is now playing a symmetry strategy. Note that the upper-right box and the lower-left box are, in fact, equivalent. If this is not obvious from the DOTS & BOXES position, try looking at the corresponding STRINGS & COINS position, where the upper-right dangling string could extend toward the right without changing the position:

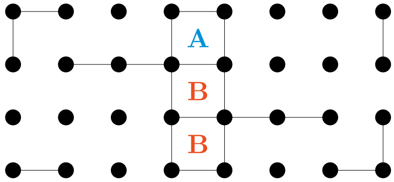




If Bob allows Alice to keep symmetry to the end of the game, Alice will succeed in getting an odd number of long chains and win. So, Bob should make a loony move now on the long chain, forcing Alice to choose between taking the whole chain or making one box and a double-dealing move:



or



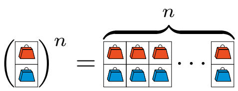
While Theorem 1.16 guarantees that Alice can win from one of the two positions (Alice first takes one box and then uses the theorem to assure half the remainder), the proof of the theorem gives no guidance about *how* to win.

If Alice chooses the first option, it is now her move on the rest of the board; she cannot play symmetry. If, on the other hand, she chooses the second option, Bob has gained a one-point advantage, which he may be able to parlay into a win.

Problems

Note that a few problems require some familiarity with graph theory. In particular, Euler's Formula, Theorem A.7 on page 263, will come in handy.

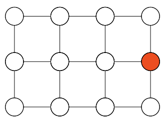
1. Consider the $2 \times n$ CLOBBER position



Show that if n is even then

is a second-player win. (By the way, the first player wins when $n \leq 13$ is odd and, we conjecture, for all n odd.)

2. Prove that Left to move can win in the COL position



3. Suppose that two players play STRINGS & COINS with the additional rule that a player, on her turn, can spend a coin to end her turn. The last player to play wins. (Spending a coin means discarding a coin that she has won earlier in the game.)

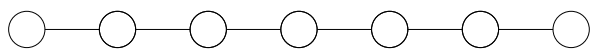
(a) Prove that the first player to take any coin wins.

(b) Suppose that the players play on an m-coin by n-coin board with the usual starting position. Prove that if $m+n$ is even, the second player can guarantee a win.

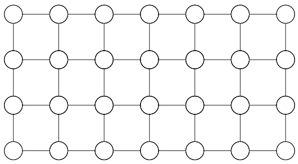
(c) Prove that if $m+n$ is odd, the first player can guarantee a win.

4. Two players play the following game on a round tabletop of radius R. Players take turns placing pennies (of unit radius) on the tabletop, but no penny is allowed to touch another or to project beyond the edge of the table. The first player who cannot legally play loses. Determine who should win as a function of R.[5]

5. Who wins SNORT when played on a path of length n?



How about an $m \times n$ grid?



6. The game of ADD-TO-15 is the same as 3-TO-15 (page 20) except that the first player to get *any* number of cards adding to 15 wins. Under perfect play, is ADD-TO-15 a first-player win, second-player win, or draw?

7. The following vertex-deletion game is played on a directed graph. A player's turn consists of removing any single vertex with even indegree (and any edges into or out of that vertex). Determine the winner if the start position is a directed tree, with all edges pointing toward the root.

8. Two players play a vertex-deletion game on an undirected graph. A turn consists of removing exactly one vertex of even degree (and all edges incident to it). Determine the winner.

9. A bunch of coins is dangling from the ceiling. The coins are tied to one another and to the ceiling by strings as pictured below. Players alternately cut strings, and a player whose cut causes any coins to drop to the ground loses. If both players play well, who wins?

[5]The players are assumed to have perfect fine motor control!